Comment on "Regression with slowly varying regressors and nonlinear trends" by P.C.B. Phillips

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Abstract

Standardized slowly varying regressors are shown to be L_p-approximable. This fact allows one to relax the assumption on linear processes imposed in central limit results by P.C.B. Phillips, as well as provide alternative proofs for some other statements.

Keywords: slowly varying regressors, central limit theorem, L_p-approximability

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Regressions with asymptotically collinear regressors have surprisingly many applications, as the references in (Phillips, 2007) show. Using the theory of slowly varying (SV) functions, Phillips has developed a method to deal with such regressions. The impact of his findings will increase if one realizes that all standardized SV regressors arising in his approach are L_p-approximable in the sense of Mynbaev (2001). We prove this fact below in Theorem 1 and apply it in Theorem 2 to generalize some central limit results established by Phillips. The corresponding functional laws will be given elsewhere. We follow the notation adopted by Phillips.

The idea will be clear from a discussion of the central limit theorem (CLT) contained in (Phillips, 2007, Eq. (9)). Under Phillips' Assumption LP, for any $f \in C^1$

$$\frac{1}{\sqrt{n}} \sum_{s=1}^{n} f\left(\frac{s}{n}\right) u_s \rightarrow_d N\left(0, \left(\sigma_\varepsilon \sum_{j=0}^{\infty} c_j\right)^2 \int_0^1 f^2(r)dr\right). \tag{1}$$

By looking at the right-hand side of this relation, one can tell that the widest class for which such convergence takes place should be L_2, the set of square-integrable functions on $(0,1)$. The CLT from (Mynbaev, 2001) is true for $f \in L_2$ (for badly behaving functions, the numbers $\frac{1}{\sqrt{n}} f\left(\frac{s}{n}\right)$ at the left of (1) should be replaced by $\sqrt{n}\int_{(s-1)/n}^{s/n} f(t)dt$). Moreover, Assumption LP can be relaxed as follows:

Assumption LP(M) $u_t = \sum_{j=-\infty}^{j=\infty} c_j e_{t-j}$, $\sum_{j=-\infty}^{j=\infty} |c_j| < \infty$, $\sum_{j=-\infty}^{j=\infty} c_j \neq 0$, with $e_t = iid(0, \sigma_e^2)$ and uniformly integrable e_t^2. (Here and in the sequel "M" stands for "modified").

Our proof of L_p-approximability derives from the proof of (Phillips, 2007, Lemma 7.4). The proof of that lemma depends on his equations (6) and (60). The limit relation (60) holds uniformly in $r \in (\delta, 1)$, where $\delta \in (0,1)$ is an arbitrary but fixed number. Condition (6) takes care of a neighborhood of 0 of type $(0, n^{-\alpha})$, $\alpha > 0$. Between $(0, n^{-\alpha})$ and $(\delta, 1)$ there is an increasing gap of $(n^{-\alpha}, \delta)$, and it is not clear from the proof of Lemma 7.4 how this gap is closed. To close a similar gap in our proof, we add to Phillips' Assumption SSV the condition that $\varepsilon(x)$ (the ε-function of L) satisfies certain monotonicity requirements.

Assumption SSV(M) (a) $L(x)$ is a smoothly slowly varying (SSV) function with Karamata representation

$$L(x) = c \exp\left(\int_a^x \frac{\varepsilon(t)}{t} dt\right) \text{ for } x \geq a \tag{2}$$

for some $a > 0$, and where $c > 0$ is a constant, $\varepsilon(x)$ is continuous and $\varepsilon(x) \to 0$ as $x \to \infty$.

(b) $|\varepsilon(x)|$ is SSV.

(c) There exists a function $\phi(x)$ on $[0, \infty)$ with properties:

(c1) ϕ is positive increasing on $[0, \infty)$, $\phi(x) \to \infty$ as $x \to \infty$, and there exist positive numbers θ, X such that $x^{-\theta}\phi(x)$ is nonincreasing on $[X, \infty)$,

(c2) $\varepsilon(x)$ is quasi-monotone in the neighborhood of ∞ in the sense that with some positive constants c_1, c_2, c_3

$$\frac{c_1}{\phi(x)} \leq |\varepsilon(x)| \leq \frac{c_2}{\phi(x)} \text{ for } x \geq c_3. \tag{3}$$

We assume that ε and L have been redefined on $[0, a]$ in such a way that L is continuous on $[0, \infty)$. Part (c) of the above assumption allows us to take advantage of the theory of SV functions with remainder due to Aljančić et al. (1955). Specifically, we utilize two facts given in the appendix of (Seneta, 1985). Theorem A.1.2 from that appendix, equation (2) and part (c) of Assumption SSV(M) imply that L is SV with remainder ϕ. Lemma A.1.1.2) from the same source states that for any $\beta > 0$ there exist numbers $M_\beta > 0$ and $B_\beta \geq a$ such that

$$\left| \frac{L(rx)}{L(x)} - 1 \right| \leq M_\beta r^{-\beta}/\phi(x) \text{ for all } x \geq B_\beta \text{ and } B_\beta/x \leq r \leq 1. \tag{4}$$

For Theorem 1 we need the following definitions. Let $p \in [1, \infty]$, $\|g\|_{p,\Omega} = \left(\int_\Omega |g(x)|^p \, dx\right)^{1/p}$ if $p < \infty$ and $\|g\|_{\infty,\Omega} = \operatorname{ess\,sup}_{x\in\Omega} |g(x)|$, where Ω is an interval. Denote L_p the space of measurable functions on $(0, 1)$ with $\|g\|_{p,(0,1)} < \infty$. A partition $i_t = [(t-1)/n, t/n)$, $t = 1, ..., n$, of the interval $[0, 1)$ generates an interpolation operator D_{np} according to

$$D_{np}w = n^{1/p} \sum_{t=1}^{n} w_t 1_{i_t}, \; w \in \mathbb{R}^n,$$

where 1_A is the indicator of a set A. We say that a sequence of vectors $\{w_n\}$, where $w_n \in \mathbb{R}^n$ for each n, is L_p-close to $g \in L_p$ if $\|D_{np}w - g\|_{p,(0,1)} \to 0$. Denote

$$G(t, n) = \frac{L(t) - L(n)}{L(n)\varepsilon(n)}, \; t = 1, ..., n.$$

Theorem 1. *For $p \in [1, \infty)$ and natural j define a vector $w_n \in \mathbb{R}^n$ by $w_{nt} = n^{-1/p} G^j(t, n)$, $t = 1, ..., n$. If Assumption SSV(M) holds and $p\theta k < 1$, then $\{w_n\}$ is L_p-close to $f_j(x) = \log^j x$.*

Of various implications of L_p-approximability we list only those directly related to (Phillips, 2007). In the next theorem references in brackets are to that paper.

Theorem 2. *Let Assumptions LP(M) and SSV(M) hold and let j be a natural number.*

(I) If $\theta k < 1$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} G^j(t,n) = (-1)^j j!$ [p.595, line 4 from bottom].

(II) If $\theta < 1$, then $\frac{1}{n}\sum_{t=1}^{n} L^j(t) = L^j(n) - jL^j(n)\varepsilon(n)[1+o(1)]$ [a weaker version of (14)].

(III) If $2\theta < 1$, then $\frac{1}{n}\sum_{t=1}^{n}(L(t)-\bar{L})^2 = L^2(n)\varepsilon^2(n)[1+o(1)]$ [p.564, line 2 from bottom].

(IV) Let $\sigma^2 = \left(\sigma_e \sum_{j=-\infty}^{\infty} c_j\right)^2$. The following central limit results are true [Lemma 2.1]:

(i) If $2\theta < 1$, then $\frac{1}{\sqrt{n}L(n)}\sum_{t=1}^{n} L(t)u_t \to_d N(0,\sigma^2)$,

(ii) If $2\theta < 1$, then $\frac{1}{\sqrt{n}L(n)\varepsilon(n)}\sum_{t=1}^{n}(L(t)-\bar{L})u_t \to_d N(0,\sigma^2)$,

(iii) If $2\theta k < 1$, then $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} G^j(t,n)u_t \to_d N(0,\sigma^2(2j)!)$.

(V) If in (Phillips, 2007, Lemma 6.1) the function $f(r,\theta)$ is just continuous over $(r,\theta)\in[0,1]\times\Theta$ and $2\theta<1$, then uniformly over $\theta \in N_n$ [equation (53)]

$$\frac{1}{\sqrt{n}L(n)}\sum_{t=1}^{n} f\left(\frac{t}{n}\right) L(t)u_t \to_d N\left(0,\sigma^2\int_0^1 f^2(r,\theta_0)du\right).$$

Remark. It can be shown that when $\sum_{j=-\infty}^{\infty} c_j = 0$ (and all other assumptions of Theorem 2 hold), convergence in distribution in (i)-(iii) and (V) is still true (and is equivalent to convergence in probability to zero).

Appendix

Proof of Theorem 1. Since $u \in i_t$ is equivalent to $t = [nu+1]$ (integer part), the equation $D_{np}w_n = \sum_{t=1}^{n} G^j(t,n)1_{i_t}$ takes a compact form $(D_{np}w_n)(u) = G^j([nu+1],n)$, $0 \le u < 1$. Let $0 < \delta \le 1/2$. For $n > n_1 = B_\beta/\delta$ the interval $(B_\beta/n, \delta)$ is nonempty and

$$\begin{aligned} \|D_{np}w_n - f_j\|_{p,(0,1)} \le\ & \|D_{np}w_n - f_j\|_{p,(\delta,1)} + \|f_j\|_{p,(0,\delta)} \\ & + \|D_{np}w_n\|_{p,(0,B_\beta/n)} + \|D_{np}w_n - f_j\|_{p,(B_\beta/n,\delta)} . \end{aligned} \tag{5}$$

Obviously, $\|f_j\|_{p,(0,\delta)} \to 0$ as $\delta \to 0$. Now we consider three cases.

Case $\delta \le u < 1$. In the proof of (Phillips, 2007, Eq. (60)) one can consider not only $r \le 1$ but also $r > 1$. Then one gets

$$G^j(rn,n) = \log^j r[1+o(1)] \text{ uniformly in } r \in \left(\delta, 1 + \frac{1}{2B_\beta}\right). \tag{6}$$

Defining $r = [nu+1]/n$, from the inequality $nu < [nu+1] \le nu+1$ we have

$$\delta \le u < \frac{[nu+1]}{n} = r \le u + \frac{1}{n} < 1 + \frac{1}{n_1} \le 1 + \frac{1}{2B_\beta} \tag{7}$$

so that

$$r = u + o(1) \text{ and } r \in \left(\delta, 1 + \frac{1}{2B_\beta}\right). \tag{8}$$

(6) and (8) lead to

$$G^j([nu+1],n) - \log^j u = o(1) \text{ uniformly in } u \in (\delta, 1) .$$

This proves that

$$\|D_{np}w_n - f_j\|_{p,(\delta,1)} \to 0,\ n \to \infty. \tag{9}$$

Case $B_\beta/n \le u < \delta$. Let $n > n_2 = \max\{n_1, 2\}$. Then (7) and the conditions $u \in [B_\beta/n, \delta)$, $n > n_2$ imply

$$\frac{B_\beta}{n} \le u < r \le u + \frac{1}{n} < \delta + \frac{1}{n_2} \le 1.$$

This means we can apply (3), (4) and (7) to get

$$\left|G^j([nu+1], n)\right| \le \left[\frac{M_\beta}{r^\beta \phi(n)|\varepsilon(n)|}\right]^j \le \left[\frac{M_\beta}{c_1}\right]^j u^{-\beta j} \text{ for } u \in [B_\beta/n, \delta).$$

Taking $\beta \in \left(0, \frac{1}{pj}\right)$ we have with new constants c_3, c_4

$$\int_{B_\beta/n}^{\delta} |D_{np} w_n|^p du \le c_3 \int_0^\delta u^{-p\beta j} du = c_4 \delta^{1-p\beta j}. \tag{10}$$

Case $0 < u < B_\beta/n$. In this case $[nu+1] \le nu + 1 < B_\beta + 1$ and $L([nu+1]) \le c$ by the assumed continuity of L. Hence, $|G([nu+1], n)| \le \frac{c}{|L(n)\varepsilon(n)|} + \frac{1}{|\epsilon(n)|}$ and by the Minkowski inequality

$$\|D_{np} w_n\|_{p,(0,B_\beta/n)}^{1/j} \le \left(\frac{c}{|L(n)\varepsilon(n)|} + \frac{1}{|\varepsilon(n)|}\right)\left(\frac{B_\beta}{n}\right)^{1/(pj)}. \tag{11}$$

Here the expression on the right tends to zero as $n \to \infty$ because any real powers and products of SV functions are SV and $n^{-\alpha} f(n) \to 0$ for any $\alpha > 0$ and SV function f.

From (9), (10) and (11) we see that we can choose first a small δ and then a large n to make the left side of (5) as small as desired. ■

Proof of Theorem 2. (I) With $p = 1$ Theorem 1 gives

$$\begin{aligned} \left|\frac{1}{n}\sum_{t=1}^{n} G^j(t,n) - (-1)^j j!\right| &= \left|\int_0^1 D_{n1} w_n du - \int_0^1 \log^j u du\right| \\ &\le \|D_{n1} w_n - f_j\|_{1,(0,1)} \to 0. \end{aligned}$$

(II) Letting $j = 1$ in (I) we have

$$\frac{1}{n}\sum_{t=1}^{n} L(t) = L(n) - L(n)\varepsilon(n)[1 + o(1)]. \tag{12}$$

If L satisfies Assumption SSV(M), then L^j also satisfies that assumption, its ε-function being $j\varepsilon(x)$. Application of (12) to L^j proves (II).

(III) Another application of (I) yields

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{L(t)-\bar{L}}{L(n)\varepsilon(n)}\right)^2 &= \frac{1}{L^2(n)\varepsilon^2(n)}\left\{\frac{1}{n}\sum_{t=1}^{n} L^2(t) - \left[\frac{1}{n}\sum_{t=1}^{n} L(t)\right]^2\right\} \\ &= \frac{1}{n}\sum_{t=1}^{n} G^2(t,n) - \left[\frac{1}{n}\sum_{t=1}^{n} G(t,n)\right]^2 \to 2 - 1 = 1.\end{aligned}$$

It remains to multiply both sides by $L^2(n)\varepsilon^2(n)$.

(IV) By (Mynbaev, 2001, Theorem 4.1) it is enough to establish that the sequence of weights $\{w_n\}$ is L_2-close to $g \in L_2$ to conclude that $\sum_{t=1}^{n} w_{nt}u_t \to_d N\left(0, \sigma^2 \int_0^1 g^2(u)du\right)$.

(i) Setting $p = 2$, $j = 1$ in Theorem 1 gives

$$\int_0^1 |G([nu+1], n) - \log u|^2\, du \to 0.$$

Multiply this relation by $\varepsilon^2(n) \to 0$ to obtain

$$\int_0^1 |L([nu+1])/L(n) - 1|^2\, du \to 0.$$

This means that the sequence $w_n = \frac{1}{\sqrt{n}L(n)}(L(1), ..., L(n))$ is L_2-close to $g \equiv 1$.

(ii) From (12) we conclude that the sequence of weights in statement (ii) is

$$
\begin{aligned}
w_n &= \frac{1}{\sqrt{n}L(n)\varepsilon(n)}(L(1)-\bar{L},...,L(n)-\bar{L}) = \\
&= \frac{1}{\sqrt{n}}(G(1,n),...,G(n,n)) + \frac{1+o(1)}{\sqrt{n}}(1,...,1).
\end{aligned}
$$

It is easy to see that the second sequence on the right is L_2-close to $g \equiv 1$. The first sequence is L_2-close to f_1 by Theorem 1. Hence, w_n is L_2-close to $g_1(x) = 1 + \log x$. The statement follows from the fact that $\int_0^1 g_1^2(u)du = 1$.

Statement (iii) follows directly from Theorem 1.

(V) Since f is uniformly continuous, the sequence $\left(f\left(\frac{1}{n},\theta\right),...,f\left(\frac{n}{n},\theta\right)\right)$ is L_∞-close to $f(r,\theta_0)$, which is a continuous function of r. By (Mynbaev, 2007, Theorem 3.3(d)) this sequence and $\frac{1}{\sqrt{n}L(n)}(L(1),...,L(n))$ (which is L_2-close to $g \equiv 1$) can be multiplied element by element to obtain a sequence $\frac{1}{\sqrt{n}L(n)}\left(f\left(\frac{1}{n},\theta\right)L(1),...,f\left(\frac{n}{n},\theta\right)L(n)\right)$ which will be L_2-close to $f(r,\theta_0)$. $\blacksquare$

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